

International Journal of Modern Physics A
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Lepton mass and mixing in a Neutrino Mass Model based on S_4 flavor symmetry

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Received Day Month Year
Revised Day Month Year

We study a neutrino mass model based on S_4 flavor symmetry which accommodates lepton mass, mixing with non-zero θ_{13} and CP violation phase. The spontaneous symmetry breaking in the model is imposed to obtain the realistic neutrino mass and mixing pattern at the tree-level with renormalizable interactions. Indeed, the neutrinos get small masses from one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one being in $\underline{2}$ and the two others in $\underline{3}$ under S_4 with both the breakings $S_4 \rightarrow S_3$ and $S_4 \rightarrow Z_3$ are taken place in charged lepton sector and $S_4 \rightarrow K$ in neutrino sector. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ in the both normal and inverted spectrum which is still missing in the neutrino mixing matrix. The relation between lepton mixing angles is also represented.

Keywords: Neutrino mass and mixing; Models beyond the standard model; Non-standard-model neutrinos, right-handed neutrinos, discrete symmetries.
PACS: 14.60.Pq; 12.60.-i; 14.60.St.

1. Introduction

The Standard Model (SM) is one of the most successful theories in the elementary particle physics, however, it leaves some unresolved issues that have been empirically verified, such as the fermion masses and mixing, the mass hierarchies problem and the CP-violating phases. It is obvious that the SM must be extended. Theoretically, there are several proposals for explanation of smallness of neutrino mass and large lepton mixing such as the Neutrino Minimal Standard Model¹⁻⁷, Two-Higgs-doublet model⁸, the scotogenic model^{a9}, the Georgi-Glashow model¹⁵, $SO(10)$ grand unification¹⁶, the texture zero models^{b17-19}, the 3-3-1 models²¹⁻²⁶ and so on.

^aDepending on the particle content, there exist models which generate an active neutrino mass at 1-loop¹⁰, 2-loop^{11,12}, or 3-loop^{13,14} level, but Ma's scotogenic model seems to be the simplest extension.

^bFor some other scenarios of this type of model, the reader can see in Ref. 20.

Among the possible extensions of SM, probably the simplest one obtained by adding right-handed neutrinos to its original structure which has been studied in Refs. 1–7. However, these extensions do not provide a natural explanation for large mass splitting between neutrinos and the lepton mixing was not explicitly explained²⁷.

There are five well-known patterns of lepton mixing²⁸, however, the Tri-bimaximal one proposed by Harrison-Perkins-Scott (HPS)^{29–32}

$$U_{\text{HPS}} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad (1)$$

seems to be the most popular and can be considered as a leading order approximation for the recent neutrino experimental data. In fact, the absolute values of the entries of the lepton mixing matrix U_{PMNS} are given in Ref. 33

$$|U_{PMNS}| = \begin{pmatrix} 0.801 \rightarrow 0.845 & 0.514 \rightarrow 0.580 & 0.137 \rightarrow 0.158 \\ 0.225 \rightarrow 0.517 & 0.441 \rightarrow 0.699 & 0.614 \rightarrow 0.793 \\ 0.246 \rightarrow 0.529 & 0.464 \rightarrow 0.713 & 0.590 \rightarrow 0.776 \end{pmatrix}. \quad (2)$$

The best fit values of neutrino mass squared differences and the leptonic mixing angles given in Ref. 33 as shown in Tabs. 1 and 2.

Table 1. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Ref.³³ for normal hierarchy.

	Best fit $\pm 1\sigma$	3σ range
$\Delta m_{21}^2 [10^{-5} \text{eV}^2]$	$7.50^{+0.19}_{-0.17}$	$7.02 \rightarrow 8.09$
$\Delta m_{31}^2 [10^{-3} \text{eV}^2]$	$2.457^{+0.047}_{-0.043}$	$2.317 \rightarrow 2.607$
$\sin^2 \theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.270 \rightarrow 0.344$
$\sin^2 \theta_{23}$	$0.452^{+0.052}_{-0.028}$	$0.382 \rightarrow 0.643$
$\sin^2 \theta_{13}$	$0.0218^{+0.0010}_{-0.0010}$	$0.0186 \rightarrow 0.0250$
$\delta [^\circ]$	306^{+39}_{-70}	$0 \rightarrow 360$

Table 2. The experimental values of neutrino mass squared splittings and leptonic mixing parameters, taken from Ref. 33 for inverted hierarchy.

	Best fit $\pm 1\sigma$	3σ range
$\Delta m_{21}^2 [10^{-5} \text{eV}^2]$	$7.50^{+0.19}_{-0.17}$	$7.02 \rightarrow 8.09$
$\Delta m_{31}^2 [10^{-3} \text{eV}^2]$	$-2.449^{+0.048}_{-0.047}$	$-2.590 \rightarrow -2.307$
$\sin^2 \theta_{12}$	$0.304^{+0.013}_{-0.012}$	$0.270 \rightarrow 0.344$
$\sin^2 \theta_{23}$	$0.579^{+0.025}_{-0.037}$	$0.389 \rightarrow 0.644$
$\sin^2 \theta_{13}$	$0.0219^{+0.0011}_{-0.0010}$	$0.0188 \rightarrow 0.0251$
$\delta [^\circ]$	254^{+63}_{-62}	$0 \rightarrow 360$

The large lepton mixing angles given in Tabs. 1, 2 are completely different from the quark mixing ones defined by the Cabibbo- Kobayashi-Maskawa (CKM) matrix^{34, 35}. This has stimulated works on flavor symmetries and non-Abelian discrete symmetries, which are considered to be the most attractive candidate to formulate dynamical principles that can lead to the flavor mixing patterns for quarks and leptons. There are various recent models based on the non-Abelian discrete symmetries, see for example A_4 ^{36–54}, S_3 ^{55–95}, S_4 ^{96–124}, D_4 ^{125–135}, T' ^{136–145}, T_7 ^{146–150}. However, in all these papers, the fermion masses and mixings generated from non-renormalizable interactions or at loop level but not at tree-level.

In this work, we investigate another choice with S_4 group, the permutation group of four objects, which is also the symmetry group of a cube. It has 24 elements divided into 5 conjugacy classes, with $\underline{1}$, $\underline{1}'$, $\underline{2}$, $\underline{3}$, and $\underline{3}'$ as its 5 irreducible representations. A brief of the theory of S_4 group is given in.¹⁵¹ We note that S_4 has not been considered before in this kind of the model in this scenario^c. This model is different from our previous works^{151–163} because the 3-3-1 models (based on $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$) itself is an extension of the SM.

The rest of this work is organized as follows. In Sec. 2 we present the necessary elements of the model and introduce necessary Higgs fields responsible for the lepton masses. Sec. 3 is devoted for the quark mass and mixing at tree level. We summarize our results and make conclusions in the section 4. Appendix A briefly provides the theory of S_4 group with its Clebsch-Gordan coefficients. Appendix B, Appendix C and Appendix D provide the breakings of S_4 by $\underline{3}$, $\underline{3}'$ and $\underline{2}$, respectively.

2. Lepton mass and mixing

The symmetry group of the model under consideration is

$$G = SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X \otimes \underline{S}_4, \quad (3)$$

where the electroweak sector of the SM is supplemented by an auxiliary symmetry $U(1)_X$ plus a S_4 flavour symmetry whereas the strong interaction one is retained. The reason for adding the auxiliary symmetry $U(1)_X$ was discussed fully in¹⁶⁴. The lepton content of the model, under $[SU(2)_L, U(1)_Y, U(1)_X, \underline{S}_4]$, is summarized in Tab. 3.

Table 3. The lepton content of the model.

Fields	$\psi_{1,2,3L}$	$l_{1(2,3)R}$	ν_R	ϕ	ϕ'	φ	χ	ζ
$SU(2)_L$	2	1	1	2	2	2	1	1
$U(1)_Y$	−1	−2	0	1	1	1	0	0
$U(1)_X$	1	1	0	0	0	−1	0	0
\underline{S}_4	$\underline{3}$	$\underline{1}(\underline{2})$	$\underline{3}$	$\underline{3}$	$\underline{3}'$	$\underline{1}$	$\underline{3}$	$\underline{2}$

^cIn this scenario, fermion masses and mixing angles are generated from renormalizable Yukawa interactions and at tree-level.

The charged lepton masses arise from the couplings of $\bar{\psi}_L l_{1R}$ and $\bar{\psi}_L l_{2R}$ to scalars, where $\bar{\psi}_L l_{1R}$ transforms as 2 under $SU(2)_L$ and $\underline{3}$ under S_4 ; $\bar{\psi}_L l_{2R}$ transforms as 2 under $SU(2)_L$ and $\underline{3} \oplus \underline{3}'$ under S_4 . To generate masses for the charged leptons, we need two scalar multiplets ϕ and ϕ' given in Tab. 3.

The Yukawa interactions are

$$-\mathcal{L}_l = h_1(\bar{\psi}_L \phi)_{\underline{1}} l_{1R} + h_2(\bar{\psi}_L \phi)_{\underline{2}} l_{2R} + h_3(\bar{\psi}_L \phi')_{\underline{2}} l_{2R} + H.c. \quad (4)$$

Theoretically, a possibility that the Tribimaximal mixing matrix (U_{HPS}) can be decomposed into only two independent rotations may provide a hint for some underlying structure in the lepton sector,

$$U_{HPS} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \end{pmatrix} \cong U_L^+ U_\nu, \quad (5)$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$.

All possible breakings of S_4 group under triplets $\underline{3}$ and $\underline{3}'$ are presented in appendices Appendix B and Appendix C, respectively. To obtain charged - lepton mixing satisfying (5), in this work we argue that both the breakings $S_4 \rightarrow S_3$ and $S_4 \rightarrow Z_3$ are taken place in charged lepton sector. The breaking $S_4 \rightarrow S_3$ can be achieved by a $SU(2)_L$ doublet ϕ with the third alignment given in Appendix B, i.e, $\langle \phi \rangle = (\langle \phi_1 \rangle, \langle \phi_1 \rangle, \langle \phi_1 \rangle)$ under S_4 , where

$$\langle \phi_1 \rangle = (0 \quad v)^T, \quad (6)$$

and the breaking $S_4 \rightarrow Z_3$ can be achieved by another $SU(2)_L$ doublet ϕ' with the third alignment given in Appendix C, i.e, $\langle \phi' \rangle = (\langle \phi'_1 \rangle, \langle \phi'_1 \rangle, \langle \phi'_1 \rangle)$ under S_4 , where

$$\langle \phi'_1 \rangle = (0 \quad v')^T. \quad (7)$$

After electroweak breaking, the mass Lagrangian for the charged leptons becomes

$$-\mathcal{L}_l^{\text{mass}} = (\bar{l}_{1L}, \bar{l}_{2L}, \bar{l}_{3L}) M_l (l_{1R}, l_{2R}, l_{3R})^T + H.c., \quad (8)$$

where

$$M_l = \begin{pmatrix} h_1 v & h_2 v - h_3 v' & h_2 v + h_3 v' \\ h_1 v & (h_2 v - h_3 v') \omega & (h_2 v + h_3 v') \omega^2 \\ h_1 v & (h_2 v - h_3 v') \omega^2 & (h_2 v + h_3 v') \omega \end{pmatrix}. \quad (9)$$

The mass matrix M_l in Eq. (9) is diagonalized by $U_L^\dagger M_l U_R = \text{diag}(m_e, m_\mu, m_\tau)$, with

$$m_e = \sqrt{3} h_1 v, \quad m_\mu = \sqrt{3} (h_2 v - h_3 v'), \quad m_\tau = \sqrt{3} (h_2 v + h_3 v'), \quad (10)$$

and^d

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad U_R = 1. \quad (11)$$

The result in Eq. (10) shows that the masses of muon and tauon are separated by the $SU(2)_L$ doublet ϕ' . This is the reason why ϕ' was additionally introduced to ϕ in lepton sector.

Now, by combining Eq. (10) with the experimental values for masses of the charged leptons given in Ref. 165,

$$m_e \simeq 0.51099 \text{ MeV}, \quad m_\mu = 105.65837 \text{ MeV}, \quad m_\tau = 1776.82 \text{ MeV} \quad (12)$$

It follows that $h_1 \ll h_2, h_3$ and $h_2 \simeq h_3$ if $v' \simeq v$. On the other hand, if we suppose that ^e $v \sim 100 \text{ GeV}$ then

$$h_1 \sim 10^{-6}, \quad h_2 \sim h_3 \sim 10^{-3}, \quad (13)$$

i.e, in the model under consideration, the hierarchy between the masses for charged-leptons can be achieved if there exists a hierarchy between Yukawa couplings h_1 and $h_{2,3}$ in charged-lepton sector as given in Eq. (13).

The neutrino masses arise from the couplings of $\bar{\psi}_L \nu_R$ and $\bar{\nu}_R^c \nu_R$ to scalars, where $\bar{\psi}_L \nu_R$ transforms as $\underline{2}$ under $SU(2)_L$ and $\underline{1} \oplus \underline{2} \oplus \underline{3}_s \oplus \underline{3}'_a$ under S_4 ; $\bar{\nu}_R^c \nu_R$ transform as $\underline{1}$ under $SU(2)_L$ and $\underline{1} \oplus \underline{2} \oplus \underline{3}_s \oplus \underline{3}'_a$ under S_4 . Note that under S_4 symmetry, each tensor product $\underline{3} \otimes \underline{3} \otimes \underline{3}$ contains one invariant^f. On the other hand, $\underline{2} \otimes \underline{2} = \underline{1} \oplus \underline{3}$ and $\underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{3} \oplus \underline{5}$ under $SU(2)_L$. For the known $SU(2)_L$ scalar doublets, only two available interactions $(\bar{\psi}_L \tilde{\phi})_{\underline{3}_s} \nu_R$, $(\bar{\psi}_L \tilde{\phi}')_{\underline{3}_a} \nu_R$, but explicitly suppressed because of the $U(1)_X$ symmetry. We therefore additionally introduce one $SU(2)_L$ doublet (φ) and two $SU(2)_L$ singlets (χ, ζ), respectively, put in $\underline{1}$, $\underline{3}$ and $\underline{2}$ under S_4 as given in Tab. 3.

It is need to note that φ contributes to the Dirac mass matrix in the neutrino sector and χ contributes to the Majorana mass matrix of the right-handed neutrinos. We also note that the $U(1)_X$ symmetry forbids the Yukawa terms of the form $(\bar{\psi}_L \tilde{\phi})_{\underline{3}_s} \nu_R$ and yield the expected results in neutrino sector, and this is interesting feature of X -symmetry.

All possible breakings of S_4 group under triplet $\underline{3}$ and doublet $\underline{2}$ are given in appendices Appendix B and Appendix D, respectively. To obtain a realistic neutrino spectrum, i.e, resulting the non-zero θ_{13} and CP violation, in this work, we argue

^dThe charged lepton mixing matrix in this model given in Eq. (11) is the same as that of in Refs.154,158,161 and a little different from that in Ref.156.

^eIn the SM, the Higgs VEV v is 246 GeV, fixed by the W boson mass and the gauge coupling, $m_W^2 = \frac{g^2}{4} v_{weak}^2$. However, in the model under consideration, $M_W^2 \simeq \frac{g^2}{2} (3v^2 + 3v'^2)$. Therefore, we can identify $v_{weak}^2 = 6(v^2 + v'^2) = (246 \text{ GeV})^2$ and then obtain $v' \simeq v \simeq 71 \text{ GeV}$. In this work, we chose $v = 100 \text{ GeV}$ for its scale.

^fIn fact $\underline{3} \otimes \underline{3}' \otimes \underline{3}$ has one invariant but this invariant vanishes in neutrino sector since $(\underline{3} \otimes \underline{3}')_{\underline{3}_a}$ contains $\underline{3}_a(23 - 32, 31 - 13, 12 - 21)$ under S_4 .

that the breaking $S_4 \rightarrow \mathcal{K}$ must be taken place in neutrino sector. This can be achieved within each case below.

- (1) A $SU(2)_L$ doublet χ put in $\underline{3}$ under S_4 with the VEV is chosen by

$$\langle \chi_1 \rangle = v_\chi, \quad \langle \chi_2 \rangle = \langle \chi_3 \rangle = 0. \quad (14)$$

- (2) Another $SU(2)_L$ doublet ζ put in $\underline{2}$ under S_4 with the VEV given by

$$\langle \zeta \rangle = (\langle \zeta_1 \rangle, \langle \zeta_2 \rangle), \quad \langle \zeta_i \rangle = v_{\zeta_i} \quad (i = 1, 2). \quad (15)$$

The Yukawa Lagrangian invariant under G symmetry in neutrino sector reads:

$$-\mathcal{L}_\nu = \frac{x}{2}(\bar{\psi}_L \tilde{\varphi})_{\underline{3}} \nu_R + \frac{y}{2}(\bar{\nu}_R^c \chi)_{\underline{3}_s} \nu_R + \frac{M}{2} \bar{\nu}_R^c \nu_R + \frac{z}{2}(\bar{\nu}_R^c \zeta)_{\underline{3}} \nu_R + H.c., \quad (16)$$

where M is the bare Majorana mass for the right-handed neutrino.

After electroweak breaking, the mass Lagrangian for the neutrinos is given by

$$-\mathcal{L}_\nu^{mass} = \frac{1}{2} \bar{\chi}_L^c M_\nu \chi_L + H.c., \quad (17)$$

where

$$\begin{aligned} \chi_L &\equiv (\nu_L \quad \nu_R^c)^T, \quad M_\nu \equiv \begin{pmatrix} 0 & M_D \\ M_D & M_R \end{pmatrix}, \\ \nu_L &= (\nu_{1L} \quad \nu_{2L} \quad \nu_{3L})^T, \quad \nu_R^c = (\nu_{1R}^c \quad \nu_{2R}^c \quad \nu_{3R}^c)^T, \end{aligned} \quad (18)$$

and the mass matrices M_D, M_R are then obtained by

$$\begin{aligned} M_D &= m_D \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_R = \begin{pmatrix} M + M_1 + M_2 & 0 & 0 \\ 0 & M + \omega M_1 + \omega^2 M_2 & M' \\ 0 & M' & M + \omega^2 M_1 + \omega M_2 \end{pmatrix}, \\ M' &= y v_\chi, \quad m_D = x v_\varphi, \quad M_i = z v_{\zeta_i} \quad (i = 1, 2), \end{aligned} \quad (19)$$

with $v_\varphi = \langle \varphi \rangle$, and M_D is the Dirac neutrino mass matrix, M_R is the right-handed Majorana neutrino mass matrix.

The effective neutrino mass matrix, in the framework of seesaw mechanism, is given by

$$M_{\text{eff}} = -M_D^T M_R^{-1} M_D = \begin{pmatrix} A & 0 & 0 \\ 0 & B_1 & C \\ 0 & C & B_2 \end{pmatrix}, \quad (20)$$

where

$$\begin{aligned} A &= -\frac{m_D^2}{M + M_1 + M_2}, \quad B_{1,2} = \frac{m_D^2 [-2M + M_1 + M_2 \pm i\sqrt{3}(M_1 - M_2)]}{2\mathfrak{M}}, \\ C &= \frac{m_D^2 M'}{\mathfrak{M}}, \quad \mathfrak{M} = M^2 + M_1^2 + M_2^2 - M M_1 - M M_2 - M_1 M_2 - M'^2. \end{aligned} \quad (21)$$

The matrix M_{eff} in (20) can be diagonalized as follows $U_\nu^T M_{\text{eff}} U_\nu = \text{diag}(m_1, m_2, m_3)$, with

$$\begin{aligned} m_1 &= \frac{1}{2} \left(B_1 + B_2 + \sqrt{(B_1 + B_2)^2 + 4C^2} \right), & m_2 &= A, \\ m_3 &= \frac{1}{2} \left(B_1 + B_2 - \sqrt{(B_1 + B_2)^2 + 4C^2} \right), \end{aligned} \quad (22)$$

and

$$U_\nu = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{K^2+1}} & 0 & \frac{K}{\sqrt{K^2+1}} \\ -\frac{K}{\sqrt{K^2+1}} & 0 & \frac{1}{\sqrt{K^2+1}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad (23)$$

$$K = \frac{B_1 - B_2 - \sqrt{(B_1 - B_2)^2 + 4C^2}}{2C}. \quad (24)$$

The lepton mixing matrix, obtained from the matrices U_ν and U_L in Eqs. (11) and (23), is expressed as

$$U = U_L^\dagger U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} \frac{1-K}{\sqrt{K^2+1}} & 1 & \frac{1+K}{\sqrt{K^2+1}} \\ \frac{\omega(\omega-K)}{\sqrt{K^2+1}} & 1 & \frac{\omega(1+K\omega)}{\sqrt{K^2+1}} \\ \frac{\omega(1-K\omega)}{\sqrt{K^2+1}} & 1 & \frac{\omega(\omega+K)}{\sqrt{K^2+1}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & i \end{pmatrix}. \quad (25)$$

where K is defined in Eq.(24).

In the standard parametrization, the lepton mixing matrix can be parametrized as¹⁶⁵

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{13} & -s_{13}e^{-i\delta} \\ s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta} & -s_{23}c_{13} \\ s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \times P, \quad (26)$$

where $P = \text{diag}(1, e^{i\alpha}, e^{i\beta})$, and $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ with θ_{12} , θ_{23} and θ_{13} being the solar, atmospheric and reactor angles, respectively, and $\delta = [0, 2\pi]$ is the Dirac CP violation phase while α and β are two Majorana CP violation phases.

Comparing the lepton mixing matrix in Eq. (25) to the standard parametrization in Eq.(26), one obtains $\alpha = 0, \beta = \pi/2$, and

$$s_{13}e^{-i\delta} = \frac{-1-K}{\sqrt{3}\sqrt{K^2+1}}, \quad (27)$$

$$t_{12} = \frac{\sqrt{K^2+1}}{K-1}, \quad (28)$$

$$t_{23} = -\frac{1+K\omega}{K+\omega}. \quad (29)$$

Substituting $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ into Eq. (29) yields:

$$\text{Re}K = \frac{t_{23}^2 - 4t_{23} + 1}{2(t_{23}^2 - t_{23} + 1)}, \quad \text{Im}K = \frac{\sqrt{3}}{2} \frac{1 - t_{23}^2}{t_{23}^2 - t_{23} + 1}. \quad (30)$$

It is easily to see that $|K| = \sqrt{(ImK)^2 + (ReK)^2} = 1$. Combining Eq. (27) and Eq. (28) we obtain:

$$e^{-i\delta} = \frac{1}{\sqrt{3}s_{13}t_{12}} \frac{1+K}{1-K}.$$

or

$$-i \frac{t_{23} - 1}{s_{13}t_{12}(t_{23} + 1)} = \cos \delta - i \sin \delta. \quad (31)$$

By equating the real and imaginary parts of the equation (31), we get

$$\cos \delta = 0, \quad \sin \delta = \frac{t_{23} - 1}{s_{13}t_{12}(t_{23} + 1)}. \quad (32)$$

Since $\cos \delta = 0$ so that $\sin \delta$ must be equal to ± 1 , it is then $\delta = \frac{\pi}{2}$ or $\delta = -\frac{\pi}{2}$. The value of the Jarlskog invariant J_{CP} which determines the magnitude of CP violation in neutrino oscillations is determined¹⁶⁵

$$J_{CP} = \frac{1}{8} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \sin \delta. \quad (33)$$

Once θ_{12} , θ_{23} and θ_{13} have been determined experimentally, the size of J_{CP} depends essentially only on the magnitude of the currently not well determined value of the Dirac phase δ . Thus, our model predicts the maximal Dirac CP violating phase which is the same as in Refs. 166,167 but the difference comes from θ_{23} . Namely, in Refs. 166,167 $\theta_{23} = \pi/4$ but in our model $\theta_{23} \neq \pi/4$ which is more consistent with the recent experimental data given in Tabs. 1, 2 and this is one of the most striking prediction of the model under consideration.

At present, the precise evaluation of θ_{23} is still an open problem while θ_{12} and θ_{13} are now very constrained¹⁶⁵. From Eq. (32), as will see below, our model can provide constraints on θ_{23} from θ_{12} and θ_{13} which satisfy the data given in Ref.165.

- (i) In the case $\delta = \frac{\pi}{2}$, from (32) we have the relation among three Euler's angles as follows:

$$t_{23} = \frac{1 + s_{13}t_{12}}{1 - s_{13}t_{12}}, \quad (34)$$

or

$$s_{23}^2 = \frac{(1 - s_{12}^2) \left(1 + \sqrt{\frac{s_{12}^2 s_{13}^2}{1 - s_{12}^2}}\right)^2}{2[1 + s_{12}^2(s_{13}^2 - 1)]}. \quad (35)$$

In Fig. 1, we have plotted the values of s_{23}^2 as a function of s_{12}^2 and s_{13}^2 with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ given in Ref. 33 in the case $\delta = \frac{\pi}{2}$ at the 3σ level.

Taking the new data $s_{12}^2 = 0.30$ ($\theta_{12} = 33.46^\circ$) and $s_{13}^2 = 0.0245$ ($\theta_{13} = 9.00^\circ$) we obtain $s_{23}^2 = 0.6014$, i.e, $\theta_{23} = 50.8507^\circ$ which is larger than 45° , and

$$K = -0.938924 - 0.344125i, \quad (|K| = 1). \quad (36)$$

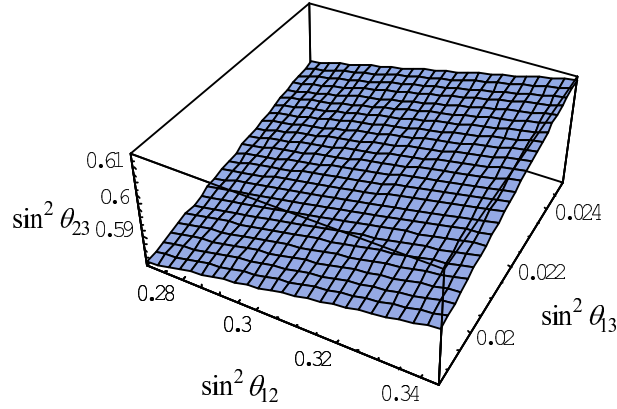


Fig. 1. s_{23}^2 as a function of s_{12}^2 and s_{13}^2 with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ in the case $\delta = \frac{\pi}{2}$ at the 3σ level.

The lepton mixing matrix in (25) then takes the form

$$U \simeq \begin{pmatrix} 0.82841 & 0.57735 & -0.147252 \\ -0.53546 & 0.57735 & -0.78743 \\ -0.29295 & 0.57735 & 0.64742 \end{pmatrix}, \quad (37)$$

which is consistent with constraint in Eq. (2).

Combining (24) and the values of K in (36), we obtain the relation

$$B_1 = B_2 - (2.75481 \times 10^{-7} + 0.68825i)C. \quad (38)$$

- (ii) Similar to the case with $\delta = \frac{\pi}{2}$, in the case $\delta = -\frac{\pi}{2}$, we find the followings relation:

$$s_{23}^2 = \frac{(1 - s_{12}^2) \left(-1 + \sqrt{\frac{s_{12}^2 s_{13}^2}{1 - s_{12}^2}} \right)^2}{2[1 + s_{12}^2(s_{13}^2 - 1)]}. \quad (39)$$

In Fig. 2, we have plotted the values of s_{23}^2 as a function of s_{12}^2 and s_{13}^2 with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ given in Ref. 33 in the case $\delta = -\frac{\pi}{2}$ at the 3σ level.

If $s_{12}^2 = 0.30$ and $s_{13}^2 = 0.0245$ we obtain $s_{23}^2 = 0.39860$ ($\theta_{23} = 39.15^\circ$), and

$$K = -0.938924 + 0.344125i, \quad (|K| = 1). \quad (40)$$

In this case the lepton mixing matrix in (25) takes the form:

$$U \simeq \begin{pmatrix} 0.82967 & 0.57735 & -0.14725 \\ -0.28731 & 0.57735 & -0.64489 \\ -0.54236 & 0.57735 & 0.79214 \end{pmatrix}, \quad (41)$$

The relation between $B_{1,2}$ and C is determined as follows

$$B_1 = B_2 - (2.75481 \times 10^{-7} - 0.68825i)C. \quad (42)$$

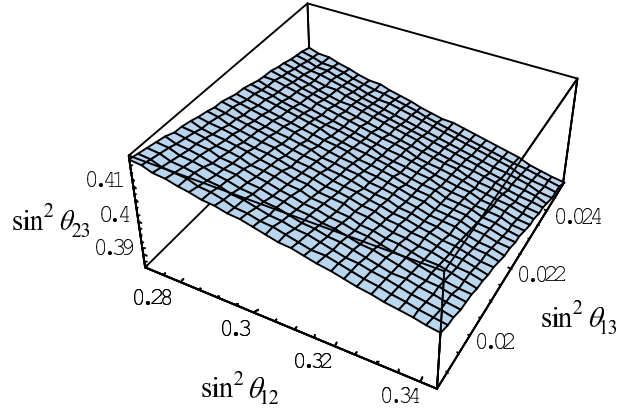


Fig. 2. s_{23}^2 as a function of s_{12}^2 and s_{13}^2 with $s_{12}^2 \in (0.270, 0.344)$, $s_{13}^2 \in (0.0186, 0.0250)$ in the case $\delta = -\frac{\pi}{2}$ at the 3σ level.

2.1. Normal case ($\Delta m_{23}^2 > 0$)

In this case, substituting B_1 from (38) into (22) and taking the two experimental data on squared mass differences of neutrinos given in Ref. 33, $\Delta m_{21}^2 = 7.50 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 = 2.457 \times 10^{-3} \text{ eV}^2$, we get a solution^g (in [eV]) as shown in Appendix Appendix E. Using the upper bound on the absolute value of neutrino mass Refs. 168–170 we can restrict the values of A , $A \leq 0.6 \text{ eV}$. However, in the case in (E.1), $|A| \in (0.00867, 0.02) \text{ eV}$ can reach the normal neutrino mass hierarchy which is depicted in Fig. 3^h.

In the model under consideration, the effective neutrino mass from tritium beta decay $m_\beta = \sqrt{\sum_{i=1}^3 |U_{ei}|^2 m_i^2}$ and the neutrino mass obtained from neutrinoless double-beta decays $m_{\beta\beta} = |\sum_{i=1}^3 U_{ei}^2 m_i|$ are depicted in Fig. 4. We also note that in the normal spectrum, $|m_1| \approx |m_2| < |m_3|$, so $m_1 \equiv m_{light}$ is the lightest neutrino mass.

To get explicit values of the model parameters, we assume $A = 10^{-2} \text{ eV}$, which is safely smallⁱ. Then the other neutrino masses are explicitly given as

$$m_1 = -5.00 \times 10^{-3} \text{ eV}, m_2 = 10^{-2} \text{ eV}, m_3 \simeq -4.982 \times 10^{-2} \text{ eV}, \quad (43)$$

$$m_{\beta\beta} = 1.88866 \times 10^{-3} \text{ eV}, m_\beta = 1.02156 \times 10^{-2} \text{ eV}, \quad (44)$$

$$|m_1| + |m_2| + |m_3| = 6.48197 \times 10^{-2} \text{ eV}, \quad (45)$$

^gThe system of equations has two solutions but they have the same absolute values of $m_{1,2,3}$, the unique difference is the sign of them. So, here we only consider in detail the case in Eq. (E.1).

^hThe expressions (E.1), (22) and (38) show that m_i ($i = 1, 2, 3$) depends only on one parameter $A \equiv m_2$ so we consider $m_{1,3}$ as functions of m_2 . However, to have an explicit hierarchy on neutrino masses m_2 should be included in the figures.

ⁱThe precise value of the mass of neutrinos is still an open question, however, it lies in the range of a few eV.

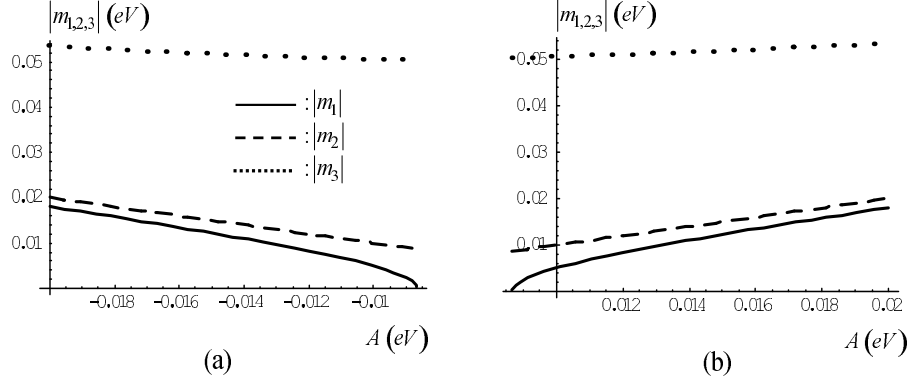


Fig. 3. $|m_{1,2,3}|$ as functions of A in the normal hierarchy with a) $A \in (-0.02, -0.00867)$ eV and b) $A \in (0.00867, 0.02)$ eV.

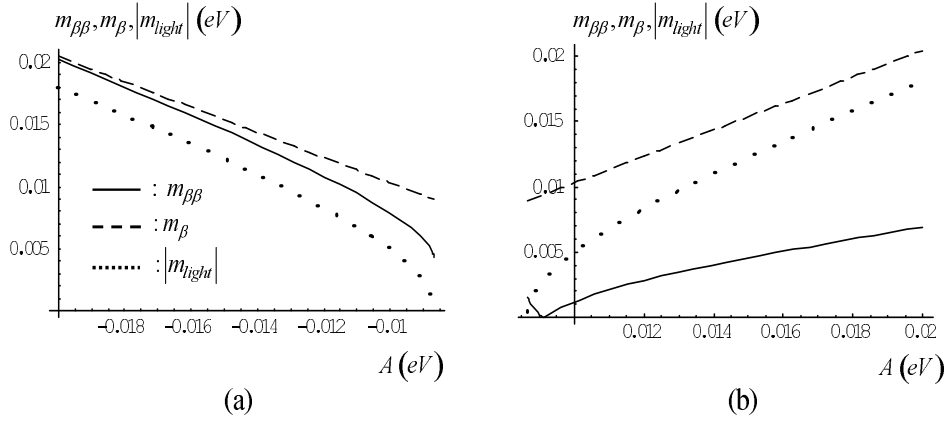


Fig. 4. m_{β} , $m_{\beta\beta}$ and $|m_{light}|$ as functions of A in the normal hierarchy with a) $A \in (-0.02, -0.00867)$ eV and b) $A \in (0.00867, 0.02)$ eV.

and

$$\begin{aligned} B_{1,2} &= -(2.74098 \pm 0.821343i) \times 10^{-2} \text{ eV}, \\ C &= (2.38676 - 1.28329i) \times 10^{-2} \text{ eV} \simeq 2.38676 \times 10^{-2} \text{ eV}. \end{aligned} \quad (46)$$

Furthermore, combining Eqs. (21) and (46) we get a solution^j:

$$\begin{aligned} M' &= (2.39395 - 1.28716 \times 10^{-7}i)M, \quad m_D = (-0.158066 + 3.21117 \times 10^{-17}i)\sqrt{M}, \\ M_{1,2} &= (-2.22488 \pm 2.15951 \times 10^{-7}i)M. \end{aligned} \quad (47)$$

^jThis system of equations has two solutions, however, these solutions differ only by the sign of m_D (or the sign of $m_{1,2,3}$) which has no effect on the neutrino oscillation experiments.

and

$$\begin{aligned} x &= (-0.158066 + 3.21117 \times 10^{-17}i)\sqrt{M}/v_\varphi \simeq -0.158066\sqrt{M}/v_\varphi, \\ y &= (2.39395 - 1.28716 \times 10^{-7}i)M/v_\chi \simeq 2.39395M/v_\chi, \\ z &= (-2.22488 + 2.15951 \times 10^{-7}i)M/v_{\zeta_1} \simeq -2.22488M/v_1, \end{aligned} \quad (48)$$

$$v_{\zeta_2} = (0.572442 + 1.52625 \times 10^{-7}v_{\zeta_1}) \simeq 0.572442v_{\zeta_1}. \quad (49)$$

Eq. (49) shows that v_{ζ_1} and v_{ζ_2} are different from each other but in the same order of magnitude^k. The solution in Eq. (43) constitutes the normal spectrum and consistent with the constraints on the absolute value of the neutrino masses.^{33, 165, 170}

Similarly, in the case $\delta = -\frac{\pi}{2}$, the numerical fit of all parameters to lepton mass and mixing data is summarized in Tab. 4.

Table 4. The observables and parameters of the model in the case $\delta = -\pi/2$.

Observables	Data fit 3σ range from Ref. 33	The values of the model parameters
$\theta_{12}(^{\circ})$	$31.29 \rightarrow 35.91$	33.46
$\theta_{23}(^{\circ})$	$38.2 \rightarrow 53.3$	39.15
$\theta_{13}(^{\circ})$	$7.87 \rightarrow 9.11$	9.0
Δm_{21}^2	$(7.02 \rightarrow 8.09) \times 10^{-5} \text{ eV}^2$	7.50
Δm_{31}^2	$(2.317 \rightarrow 2.607) \times 10^{-3} \text{ eV}^2$	2.457
$ m_1 [\text{eV}]$	—	5×10^{-3}
$ m_2 [\text{eV}]$	—	10^{-2}
$ m_3 [\text{eV}]$	—	5.05668×10^{-2}
$\sum m_i [\text{eV}]$	—	4.55668×10^{-2}
$m_{\beta\beta} [\text{eV}]$	—	1.20486×10^{-3}
$m_\beta [\text{eV}]$	—	1.02949×10^{-3}
$A [\text{eV}]$	—	10^{-2}
$B_{1,2} [\text{eV}]$	—	$(-2.77834 \pm 0.835034i) \times 10^{-2}$
$C [\text{eV}]$	—	2.42654×10^{-2}

The parameters x, y, z are given as follows:

$$x \simeq -0.158066\sqrt{M}/v_\varphi, \quad y \simeq 2.40384M/v_\chi, \quad z \simeq 1.27474M/v_1, \quad (50)$$

$$v_{\zeta_2} \simeq 1.74932v_{\zeta_1}. \quad (51)$$

2.2. Inverted case ($\Delta m_{32}^2 < 0$)

By taking the two experimental data on squared mass differences of neutrinos for the inverted hierarchy given in Ref.,³³ $\Delta m_{21}^2 = 7.50 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 = -2.449 \times 10^{-3} \text{ eV}^2$, we obtain the relations^l between $m_{1,3}$ and $m_2 = A$ as shown in Fig. 5.

^kIn the case $v_{\zeta_1} = v_{\zeta_2}$, i.e, $M_1 = M_2$, the lepton mixing matrix U_{lep} in Eq. (25) becomes an exact Tri-bimaximal mixing which can be considered as a good approximation for the recent neutrino experimental data. Hence, the condition $v_{\zeta_1} \neq v_{\zeta_2}$ is necessary to reach the realistic neutrino spectrum, and the relation (49) is satisfy this condition.

^lWe only consider here one solution with $\delta = \frac{\pi}{2}$.

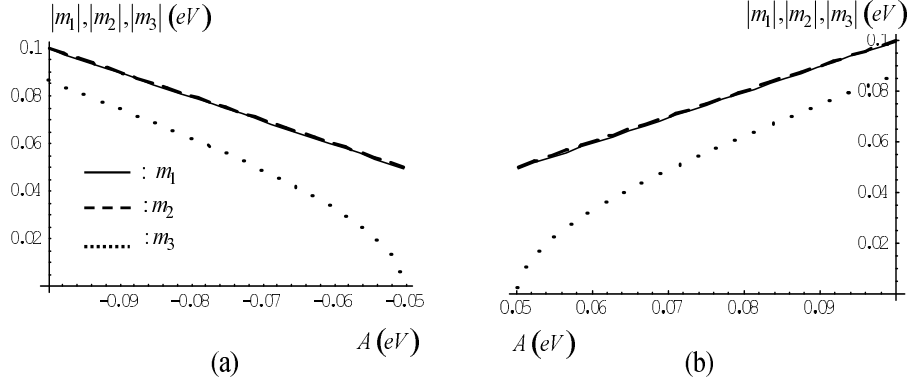


Fig. 5. $|m_{1,2,3}|$ as functions of A in the inverted hierarchy with a) $A \in (-0.1, -0.0503)$ eV and b) $A \in (0.0503, 0.1)$ eV.

In the inverted hierarchy^m, $m_3 \equiv m_{light}^I$ is the lightest neutrino mass, and the effective neutrino mass from tritium beta decay and the neutrino mass obtained from neutrinoless double-beta decays are plotted in Fig. 6.

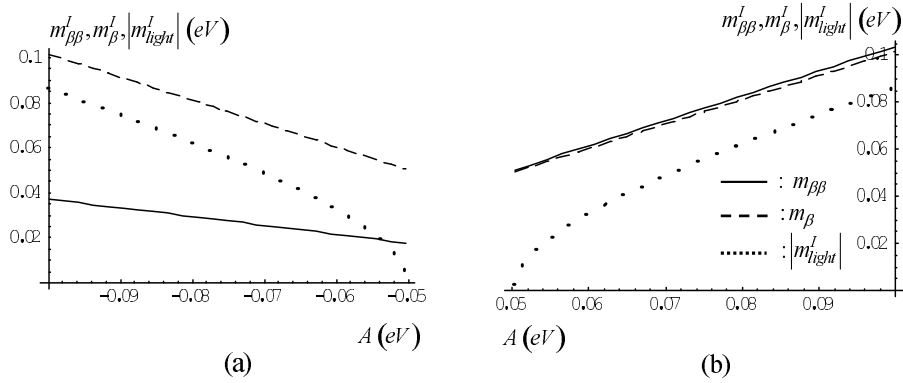


Fig. 6. m_{β}^I , $m_{\beta\beta}^I$ and $|m_{light}^I|$ as functions of A in the normal hierarchy with a) $A \in (-0.1, -0.0503)$ eV and b) $A \in (0.0503, 0.1)$ eV.

With $A = 5.1 \times 10^{-2}$ eV, we get explicit values of the model parameters as follows:

$$m_1 \simeq 5.026 \times 10^{-2} \text{ eV}, \quad m_2 = 5.1 \times 10^{-2} \text{ eV}, \quad m_3 \simeq 8.775 \times 10^{-3} \text{ eV}, \quad (52)$$

$$m_{\beta\beta}^I \simeq 5.1786 \times 10^{-2} \text{ eV}, \quad m_{\beta}^I \simeq 5.1063 \times 10^{-2} \text{ eV}, \quad \sum^I \simeq 0.11003 \text{ eV}, \quad (53)$$

^mIn the inverted spectrum, $m_3 \sim m_2 \ll m_1$ hence m_3 can be considered as the lightest neutrino mass.

14

and

$$B_{1,2} = (2.95171 \mp 0.760222i) \times 10^{-2} \text{ eV}, \quad C \simeq 2.20914 \times 10^{-2} \text{ eV}. \quad (54)$$

Now, combining (21) and (54) yields ⁿ:

$$\begin{aligned} M' &= -0.979204M, \quad m_D = 0.139816i\sqrt{M}, \\ M_1 &= -0.1138M, \quad M_2 = -0.502898M, \end{aligned} \quad (55)$$

and

$$x = 0.139816\sqrt{M}/v_\varphi, \quad y = -0.979204M/v_\chi, \quad z = -0.1138M/v_{\zeta_1}, \quad (56)$$

$$v_{\zeta_2} = 4.41914v_{\zeta_1}. \quad (57)$$

Eq. (57) shows that v_{ζ_1} and v_{ζ_2} are different from each other but in the same order of magnitude.

3. Quark mass

The quarks content of the model under $[\text{SU}(2)_L, \text{U}(1)_Y, \text{U}(1)_X, \underline{S}_4]$ symmetries, respectively, given in Tab. 5, where $i = 1, 2, 3$ is a family index of three lepton families, which are in order defined as the components of the $\underline{3}$ representations under S_4 .

Table 5. The quark content of the model.

Fields	Q_{iL}	u_{1R}	$u_{2,3R}$	d_{1R}	$d_{2,3R}$
$\text{SU}(2)_L$	2	1	1	1	1
$\text{U}(1)_Y$	1/3	4/3	4/3	-2/3	-2/3
$\text{U}(1)_X$	0	0	0	0	0
\underline{S}_4	$\underline{3}$	$\underline{1}$	$\underline{2}$	$\underline{1}$	$\underline{2}$

The Yukawa interactions are ^o:

$$\begin{aligned} -\mathcal{L}_q &= h_1^u(\bar{Q}_{iL}\tilde{\phi})_{\underline{1}}u_{1R} + h^u(\bar{Q}_{iL}\tilde{\phi})_{\underline{2}}u_R + h'^u(\bar{Q}_{iL}\tilde{\phi}')_{\underline{2}}u_R \\ &+ h_1^d(\bar{Q}_{iL}\phi)_{\underline{1}}d_{1R} + h^d(\bar{Q}_{iL}\phi)_{\underline{2}}d_R + h'^d(\bar{Q}_{iL}\phi')_{\underline{2}}d_R + H.c. \end{aligned} \quad (58)$$

With the VEV alignments of ϕ and ϕ' as given in Eqs. (6) and (7), the mass Lagrangian of quarks reads

$$\begin{aligned} -\mathcal{L}_q^{mass} &= (\bar{u}_{1L}, \bar{u}_{2L}, \bar{u}_{3L})M_u(u_{1R}, u_{2R}, u_{3R})^T + (\bar{d}_{1L}, \bar{d}_{2L}, \bar{d}_{3L})M_d(d_{1R}, d_{2R}, d_{3R})^T \\ &+ H.c, \end{aligned} \quad (59)$$

ⁿThis system of equations has two solutions, however, these solutions differ only by the sign of m_D which has no effect in the neutrino oscillation experiments.

^oHere, $\tilde{\phi} = i\sigma_2\phi^* = \begin{pmatrix} \phi_2^0 \\ -\phi_1^- \end{pmatrix} \sim [2, -1, 0, \underline{3}]$, and $\tilde{\phi}' \sim [2, -1, 0, \underline{3}']$.

where the mass matrices for up-and down-quarks are, respectively, obtained as follows

$$M_u = \begin{pmatrix} h_1^u v & h^u v - h'^u v' & h^u v + h'^u v' \\ h_1^u v & (h^u v - h'^u v')\omega & (h^u v + h'^u v')\omega^2 \\ h_1^u v & (h^u v - h'^u v')\omega^2 & (h^u v + h'^u v')\omega \end{pmatrix}, \quad (60)$$

$$M_d = \begin{pmatrix} h_1^d v & h^d v - h'^d v' & h^d v + h'^d v' \\ h_1^d v & (h^d v - h'^d v')\omega & (h^d v + h'^d v')\omega^2 \\ h_1^d v & (h^d v - h'^d v')\omega^2 & (h^d v + h'^d v')\omega \end{pmatrix}. \quad (61)$$

The structure of the up- and down-quark mass matrices in Eqs. (60) and (61) are similar to those in Ref. 171, i.e, in the model under consideration there is no CP violation in the quark sector. The matrices M_u and M_d in Eqs. (60), (61) are, respectively, diagonalized as

$$U_L^{u+} M_u U_R^u = \text{diag} \left(\sqrt{3} h_1^u v, \sqrt{3} (h^u v - h'^u v'), \sqrt{3} (h^u v + h'^u v') \right) \\ \equiv \text{diag} (m_u, m_c, m_t), \quad (62)$$

$$U_L^{d+} M_d U_R^d = \text{diag} \left(\sqrt{3} h_1^d v, \sqrt{3} (h^d v - h'^d v'), \sqrt{3} (h^d v + h'^d v') \right) \\ \equiv \text{diag} (m_d, m_s, m_b), \quad (63)$$

where $U_L^u = U_L^d = U_L$, with U_L given in (11), are the unitary matrices, which couple the left-handed up- and down-quarks to those in the mass bases, respectively, and $U_R^u = U_R^d = 1$. Therefore, in this case, we get the quark mixing matrix

$$U_{\text{CKM}} = U_L^{d\dagger} U_L^u = 1. \quad (64)$$

This is the common property for some models based on discrete symmetry groups^{151–159,161} and can be seen as an important result of the paper since the experimental quark mixing matrix is close to the unit matrix. A small permutations such as a violation of S_4 symmetry due to unnormal Yukawa interactions will possibly providing the desirable quark mixing pattern¹⁶⁰. A detailed study on this problem is out of the scope of this work and should be skip.

In similarity to the charged leptons, the masses of pairs (c, t) and (s, b) quarks are also separated by the ϕ' scalar. The up and down quark masses are $m_u = \sqrt{3} h_1^u v$, $m_c = \sqrt{3} (h^u v - h'^u v')$, $m_t = \sqrt{3} (h^u v + h'^u v')$, $m_d = \sqrt{3} h_1^d v$, $m_s = \sqrt{3} (h^d v - h'^d v')$, $m_b = \sqrt{3} (h^d v + h'^d v')$. i.e,

$$\frac{m_u}{m_d} = \frac{h_1^u}{h_1^d}, \quad \frac{m_c}{m_s} = \frac{h^u v - h'^u v'}{h^d v - h'^d v'}, \quad \frac{m_t}{m_b} = \frac{h^u v + h'^u v'}{h^d v + h'^d v'}. \quad (65)$$

The current mass values for the quarks are given by:¹⁶⁵

$$m_u = 2.3_{-0.5}^{+0.7} \text{ MeV}, \quad m_c = 1.275 \pm 0.025 \text{ GeV}, \quad m_t = 173.21 \pm 0.51 \pm 0.71 \text{ GeV}, \\ m_d = 4.8_{-0.3}^{+0.5} \text{ MeV}, \quad m_s = 95 \pm 5 \text{ MeV}, \quad m_b = 4.18 \pm 0.03 \text{ GeV}. \quad (66)$$

With the help of Eqs. (62), (63) and (66) we obtain the followings relations:

$$\begin{aligned} h^u &= \frac{5.03695 \times 10^{10}}{v}, & h^d &= \frac{1.23409 \times 10^9}{v}, \\ h'^u &= \frac{4.96334 \times 10^{10}}{v'}, & h'^d &= \frac{1.17924 \times 10^9}{v'}, \\ h_1^u &= \frac{1.32791 \times 10^6}{v}, & h_1^d &= \frac{2.77128 \times 10^6}{v}, \end{aligned} \quad (67)$$

or

$$h^u/h^d \simeq 40, \quad h'^u/h'^d \simeq 42, \quad h_1^d/h_1^u \simeq 2, \quad (68)$$

$$h^u/h_1^u \simeq 3.8 \times 10^4, \quad h^d/h_1^d \simeq 4.5 \times 10^2, \quad (69)$$

i.e, h_1^u and h_1^d are in the same order but h^u (h'^u) is one magnitude order larger than h^d (h'^d). On the other hand, in the case $|v| \sim |v'|$ we get $h'^u/h_1^u \simeq 3.7 \times 10^4$, $h'^d/h_1^d \simeq 4.2 \times 10^2$.

To get explicit values of the Yukawa couplings in the quark sector, we assume $v' \sim v \sim 100$ GeV then

$$\begin{aligned} h^u &= 0.503695, & h'^u &= 0.496334, & h_1^u &= 1.32791 \times 10^{-5}, \\ h^d &= 1.23409 \times 10^{-2}, & h'^d &= 1.17924 \times 10^{-2}, & h_1^d &= 2.77128 \times 10^{-5}. \end{aligned} \quad (70)$$

We note that, the quarks mixing matrix in Eq. (64) has no predictive power for quarks mixing but their masses are consistent with the recent experimental data.

4. Conclusions

We have proposed a neutrino mass model based on S_4 flavor symmetry which accommodates lepton mass, mixing with non-zero θ_{13} and CP violation phase, and the quark mixing matrix is unity at tree level. The realistic neutrino mass and mixing pattern obtained at the tree-level with renormalizable interactions by one $SU(2)_L$ doublet and two $SU(2)_L$ singlets in which one being in $\underline{2}$ and the two others in $\underline{3}$ under S_4 if both the breakings $S_4 \rightarrow S_3$ and $S_4 \rightarrow Z_3$ are taken place in charged lepton sector and the breaking $S_4 \rightarrow \mathcal{K}$ taken place in neutrino sector. The model also gives a remarkable prediction of Dirac CP violation $\delta_{CP} = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ in the both normal and inverted spectrum.

Appendix A. S_4 group and Clebsch-Gordan coefficients

For convenience, we will refer to some properties of S_4 .¹⁵¹ S_4 has 24 elements divided into 5 conjugacy classes, with $\underline{1}$, $\underline{1}'$, $\underline{2}$, $\underline{3}$, and $\underline{3}'$ as its 5 irreducible representations. Any element of S_4 can be formed by multiplication of the generators S and T obeying the relations $S^4 = T^3 = 1$, $ST^2S = T$. In this paper, we work in the basis

where $\underline{3}, \underline{3}'$ are real representations whereas $\underline{2}$ is complex. One possible choice of generators is given as follows

$$\begin{aligned}
 \underline{1} : S &= 1, & T &= 1 \\
 \underline{1}' : S &= -1, & T &= 1 \\
 \underline{2} : S &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & T &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \\
 \underline{3} : S &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & T &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
 \underline{3}' : S &= -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & T &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
 \end{aligned} \tag{A.1}$$

where $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$. All the group multiplication rules of S_4 as given below.

$$\underline{1} \otimes \underline{1} = \underline{1}(11), \quad \underline{1}' \otimes \underline{1}' = \underline{1}(11), \quad \underline{1} \otimes \underline{1}' = \underline{1}'(11), \tag{A.2}$$

$$\underline{1} \otimes \underline{2} = \underline{2}(11, 12), \quad \underline{1}' \otimes \underline{2} = \underline{2}(11, -12), \tag{A.3}$$

$$\underline{1} \otimes \underline{3} = \underline{3}(11, 12, 13), \quad \underline{1}' \otimes \underline{3} = \underline{3}'(11, 12, 13), \tag{A.4}$$

$$\underline{1} \otimes \underline{3}' = \underline{3}'(11, 12, 13), \quad \underline{1}' \otimes \underline{3}' = \underline{3}(11, 12, 13), \tag{A.5}$$

$$\underline{2} \otimes \underline{2} = \underline{1}(12 + 21) \oplus \underline{1}'(12 - 21) \oplus \underline{2}(22, 11), \tag{A.6}$$

$$\begin{aligned}
 \underline{2} \otimes \underline{3} &= \underline{3}((1+2)1, \omega(1+\omega^2)2, \omega^2(1+\omega^2)3) \\
 &\oplus \underline{3}'((1-2)1, \omega(1-\omega^2)2, \omega^2(1-\omega^2)3)
 \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 \underline{2} \otimes \underline{3}' &= \underline{3}'((1+2)1, \omega(1+\omega^2)2, \omega^2(1+\omega^2)3) \\
 &\oplus \underline{3}((1-2)1, \omega(1-\omega^2)2, \omega^2(1-\omega^2)3),
 \end{aligned} \tag{A.8}$$

$$\begin{aligned}
 \underline{3} \otimes \underline{3} &= \underline{1}(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, 11 + \omega 22 + \omega^2 33) \\
 &\oplus \underline{3}_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}'_a(23 - 32, 31 - 13, 12 - 21),
 \end{aligned} \tag{A.9}$$

$$\begin{aligned}
 \underline{3}' \otimes \underline{3}' &= \underline{1}(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, 11 + \omega 22 + \omega^2 33) \\
 &\oplus \underline{3}_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}'_a(23 - 32, 31 - 13, 12 - 21),
 \end{aligned} \tag{A.10}$$

$$\begin{aligned}
 \underline{3} \otimes \underline{3}' &= \underline{1}'(11 + 22 + 33) \oplus \underline{2}(11 + \omega^2 22 + \omega 33, -11 - \omega 22 - \omega^2 33) \\
 &\oplus \underline{3}'_s(23 + 32, 31 + 13, 12 + 21) \oplus \underline{3}_a(23 - 32, 31 - 13, 12 - 21),
 \end{aligned} \tag{A.11}$$

where the subscripts s and a respectively refer to their symmetric and anti-symmetric product combinations as explicitly pointed out. In the Eqs. (A.2) to (A.11) we have used the notation $\underline{3}(1, 2, 3)$ which means some $\underline{3}$ multiplet such as $x = (x_1, x_2, x_3) \sim \underline{3}$ or $y = (y_1, y_2, y_3) \sim \underline{3}$ and so on. Moreover, the numbered multiplets such as (\dots, ij, \dots) mean $(\dots, x_i y_j, \dots)$ where x_i and y_j are the multiplet components of different representations x and y , respectively.

The rules to conjugate the representations $\underline{1}$, $\underline{1}'$, $\underline{2}$, $\underline{3}$, and $\underline{3}'$ are given by

$$\underline{2}^*(1^*, 2^*) = \underline{2}(2^*, 1^*), \quad \underline{1}^*(1^*) = \underline{1}(1^*), \quad \underline{1}'^*(1^*) = \underline{1}'(1^*), \quad (\text{A.12})$$

$$\underline{3}^*(1^*, 2^*, 3^*) = \underline{3}(1^*, 2^*, 3^*), \quad \underline{3}'^*(1^*, 2^*, 3^*) = \underline{3}'(1^*, 2^*, 3^*), \quad (\text{A.13})$$

where, for example, $\underline{2}^*(1^*, 2^*)$ denotes some $\underline{2}^*$ multiplet of the form $(x_1^*, x_2^*) \sim \underline{2}^*$.

Appendix B. The breakings of S_4 by triplet $\underline{3}$

For triplets $\underline{3}$ we have the followings alignments:

- (1) The first alignment: $\langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.
- (2) The second alignment: $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0$ or $0 \neq \langle \phi_1 \rangle = \langle \phi_3 \rangle \neq \langle \phi_2 \rangle \neq 0$ or $0 \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2TS\}$, respectively.
- (3) The third alignment: $\langle \phi_1 \rangle = \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0$ then S_4 is broken into S_3 which consisting of the elements $\{1, T, T^2, TSTS^2, STS^2, S^2TS\}$.
- (4) The fourth alignment: $0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle = \langle \phi_3 \rangle \neq 0$ or $0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle \neq 0$ or $0 = \langle \phi_3 \rangle \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, TSTS^2\}$ or $\{1, TSS^2\}$ or $\{1, S^2TS\}$, respectively.
- (5) The fifth alignment: $0 = \langle \phi_2 \rangle \neq \langle \phi_1 \rangle \neq \langle \phi_3 \rangle \neq 0$ or $0 = \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle \neq 0$ or $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle = 0$ then S_4 is completely broken.
- (6) The sixth alignment: $0 \neq \langle \phi_1 \rangle \neq \langle \phi_2 \rangle = \langle \phi_3 \rangle = 0$ or $0 \neq \langle \phi_2 \rangle \neq \langle \phi_3 \rangle = \langle \phi_1 \rangle = 0$ or $0 \neq \langle \phi_3 \rangle \neq \langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ then S_4 is broken into Klein four group \mathcal{K} which consisting of the elements $\{1, S^2, TSTS^2, TST\}$ or $\{1, TS^2T^2, STS^2, T^2S\}$ or $\{1, T^2S^2T, ST^2, S^2TS\}$, respectively.

Appendix C. The breakings of S_4 by triplet $\underline{3}'$

For triplets $\underline{3}'$ we have the followings alignments:

- (1) The first alignment: $\langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.
- (2) The second alignment: $0 \neq \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle = \langle \phi'_3 \rangle \neq \langle \phi'_2 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle \neq 0$ then S_4 is broken into $\{1\} \equiv \{\text{identity}\}$, i.e. S_4 is completely broken.
- (3) The third alignment: $\langle \phi'_1 \rangle = \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ then S_4 is broken into Z_3 that consists of the elements $\{1, T, T^2\}$.
- (4) The fourth alignment: $0 = \langle \phi'_2 \rangle \neq \langle \phi'_1 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle = \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_3 \rangle \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle \neq 0$ then S_4 is broken into Z_2 which consisting of the elements $\{1, T^2S\}$ or $\{1, TST\}$ or $\{1, ST^2\}$, respectively.
- (5) The fifth alignment: $0 = \langle \phi'_2 \rangle \neq \langle \phi'_1 \rangle \neq \langle \phi'_3 \rangle \neq 0$ or $0 = \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle \neq 0$ or $0 \neq \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle = 0$ then S_4 is completely broken.

- (6) The sixth alignment: $0 \neq \langle \phi'_1 \rangle \neq \langle \phi'_2 \rangle = \langle \phi'_3 \rangle = 0$ or $0 \neq \langle \phi'_2 \rangle \neq \langle \phi'_3 \rangle = \langle \phi'_1 \rangle = 0$ or $0 \neq \langle \phi'_3 \rangle \neq \langle \phi'_1 \rangle = \langle \phi'_2 \rangle = 0$ then S_4 is broken into a four-element subgroup generated by a four-cycle, which consisting of the elements $\{1, S, S^2, S^3\}$ or $\{1, TST^2, ST, TS^2T^2\}$ or $\{1, TS, T^2ST, T^2S^2T\}$, respectively.

Appendix D. The breakings of S_4 by doublet 2

- (1) The first alignment: $\langle \zeta_1 \rangle = \langle \zeta_2 \rangle$ then S_4 is broken into an eight-element subgroup, which is isomorphic to D_4 .
- (2) The second alignment: $\langle \zeta_1 \rangle \neq 0 = \langle \zeta_2 \rangle$ or $\langle \zeta_1 \rangle = 0 \neq \langle \zeta_2 \rangle$ then S_4 is broken into A_4 consisting of the identity and the even permutations of four objects.
- (3) The third alignment: $\langle \zeta_1 \rangle \neq \langle \zeta_2 \rangle \neq 0$ then S_4 is broken into a four - element subgroup consisting of the identity and three double transitions, which is isomorphic to Klein four group \mathcal{K} .

Appendix E. The solution with $\delta = \frac{\pi}{2}$ in the normal case

By substituting B_1 from (38) into (22) and taking the two experimental data on squared mass differences of neutrinos given in Ref.,³³ $\Delta m_{21}^2 = 7.50 \times 10^{-5} \text{ eV}^2$ and $\Delta m_{31}^2 = 2.457 \times 10^{-3} \text{ eV}^2$, we get a solution (in [eV]) as follows:

$$\begin{aligned} C &= 0.5\sqrt{\alpha - 2\sqrt{\beta}}, \\ B_2 &= -0.5\sqrt{4A^2 - 0.0003} + (1.37741 \times 10^{-7} + 0.34412i)C \\ &\quad - 0.5\sqrt{(3.52631 + 3.792 \times 10^{-7}i)C^2}, \end{aligned} \quad (\text{E.1})$$

where

$$\alpha = (0.0026169 - 2.81407 \times 10^{-10}i) + (2.26866 - 2.43959 \times 10^{-7}i)A^2, \quad (\text{E.2})$$

$$\begin{aligned} \beta &= -2.2987 \times 10^{-7} + 4.94378 \times 10^{-14}i + (0.00296843 - 6.38415 \times 10^{-10}i)A^2 \\ &\quad + (1.2867 - 2.7673 \times 10^{-7}i)A^4. \end{aligned} \quad (\text{E.3})$$

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